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**Abstract**

This paper discusses geometric issues arising in the study of relativistic cosmology, particularly as seen by their evolution in the state-space of models. Two main approaches are via space-time symmetries, and by imposing conditions on covariantly defined variables. At present these two approaches are not satisfactorily related to each other.

## 1 Specifying models

A cosmological model represents the universe at a particular scale. It is defined by specifying (Ehlers 1961, 1993, Ellis 1971, 1973):

* the *space-time geometry* (determined by the metric), which —because of the requirement of compatibility with observations— must either have some expanding Robertson-Walker ('RW') geometries as a regular limit (see Krasinksi 1993), or else be demonstrated to have observational properties compatible with the major features of current astronomical observations of the universe;

* the *matter present* and its behaviour (the stress tensor of each matter component, the equations governing the behaviour of each such component, and the interaction terms between them), which must represent physically plausible matter; and

* the *interaction of the geometry and matter* —how matter determines the geometry, which in turn determines the motion of the matter. Usually we assume this is through the Einstein gravitational field equations ('EFE')

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}\,, \tag{1}$$

which guarantee the conservation of total energy-momentum because of the *contracted Bianchi identities*

$$G^{ab}{}_{;b} = 0 \;\Rightarrow\; T^{ab}{}_{;b} = 0\,. \tag{2}$$

The usual choices for the matter description will be

* a fluid with given equation of state, for example a perfect fluid with 4-velocity $u^a$, energy density $\mu$, and pressure $p$, where $p = p(\mu)$, $\mu + p > 0$ (beware of imperfect fluids, unless they have well-defined and motivated physical properties);

* a mixture of fluids, with the same or different 4-velocities;

* a set of particles represented by a kinetic theory description;

* a scalar field $\phi$, with a given potential $V(\phi)$ (at early times);

* possibly an electromagnetic field described by Maxwell's equations.

To be useful in an explanatory role, a cosmological model must be easy to describe —that means they have symmetries or special properties of some kind or other. However we are interested in the *full* state space of solutions, allowing us to see how more realistic models are related to each other and to higher symmetry models.

## 2 Covariant description and equations

It should be emphasized that the equations considered here are exact, generic, and describe a cosmological context.

### *2.1 Variables*

#### 2.1.1 The average 4-velocity of matter

In a cosmological space-time $(\mathcal{M}, \mathrm{g})$ there will be a family of 'fundamental observers' moving with the average motion of matter at each point. Their 4-velocity is

$$u^a = \frac{dx^a}{d\tau}, \; u^a u_a = -1 \tag{3}$$

where $\tau$ is proper time measured along the fundamental worldlines. We assume this 4-velocity is unique: that is, there is a preferred motion of matter at each space-time event. At recent times this is taken to be the 4-velocity defined by the dipole of the Cosmic Blackbody Radiation ('CBR'): for there is precisely one

4-velocity which will set this dipole to zero. It is usually assumed that this is the same as the average 4-velocity of matter in a suitably sized volume (Ellis 1971).

Given $u^a$, there are defined unique projection tensors

$$U^a_b = -u^a u_b \quad \Rightarrow \quad U^a{}_b U^b{}_c = U^a{}_c,\ U^a{}_a = 1,\ U_{ab}u^b = u_a\,, \tag{4}$$

$$h_{ab} = g_{ab} + u_a u_b \quad \Rightarrow \quad h^a{}_b h^b{}_c = h^a{}_c,\ h^a{}_a = 3,\ h_{ab}u^b = 0. \tag{5}$$

The first projects parallel to the velocity vector $u^a$, and the second determines the metric properties of the instantaneous rest-space of observers moving with 4-velocity $u^a$. There is also defined a volume element for the rest-spaces

$$\eta^{abc} = \eta^{abcd}u_d \ \Rightarrow\ \eta^{abc} = \eta^{[abc]},\ \eta^{abc}u_c = 0 \tag{6}$$

where $\eta^{abcd}$ is the 4-dimensional volume element ($\eta^{abcd} = \eta^{[abcd]}$, $\eta^{0123} = 1/\sqrt{|\det g_{ab}|}$.)

Two derivatives are also defined: the time derivative $\dot{}$ along the fundamental world lines, where for any tensor $T$

$$\dot{T}^{ab}{}_{cd} = T^{ab}{}_{cd;e}u^e\,, \tag{7}$$

and the orthogonal spatial derivative $\hat{\nabla}$, where for any tensor $T$

$$\hat{\nabla}_e T^{ab}{}_{cd} = h^a{}_s h^b{}_t h_c{}^v h_d{}^w \nabla_p T^{st}{}_{vw} h^p{}_e \tag{8}$$

with total projection on all free indices (note that we interchangeably use a semi-colon and $\nabla_a$ for the covariant derivative: $T^a{}_{b;c} \equiv \nabla_c T^a{}_b$).

### 2.1.2 Kinematic quantities

We split the first covariant derivative of $u_a$ into its irreducible parts, defined by their symmetry properties:

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - \dot{u}_a u_b \tag{9}$$

where $\omega_{ab}$ is the vorticity tensor ($\omega_{ab} = \omega_{[ab]}$, $\omega_{ab}u^b = 0$), $\sigma_{ab}$ is the shear tensor ($\sigma_{ab} = \sigma_{(ab)}$, $\sigma_{ab}u^b = 0$, $\sigma^a{}_a = 0$), $\Theta = u^a{}_{;a} = 3H$ is the (volume) expansion (and $H$ the Hubble parameter), and $\dot{u}_a = u_{a;b}u^b$ is the acceleration.

### 2.1.3 Matter tensor

The matter stress tensor can be decomposed relative to $u^a$ in the form

$$T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab}, \tag{10}$$
$$q_a u^a = 0, \; \pi_{ab} = \pi_{ba}, \; \pi_{ab} u^b = 0, \; \pi^a{}_a = 0$$

where $\mu = T_{ab} u^a u^b$ is the relativistic energy density, $q_a = -T_{ab} u^b$ is the relativistic momentum density, which is also the energy flux relative to $u^a$, $p = \frac{1}{3} T^a{}_a$ is the isotropic pressure, and $\pi_{ab}$ is the trace-free anisotropic stresses.

The physics of the situation is in the equations of state relating these quantities, for example the commonly imposed restrictions

$$q_a = 0 = \pi_{ab} \;\Leftrightarrow\; T_{ab} = \mu u_a u_b + p h_{ab} \tag{11}$$

characterize a 'perfect fluid'. If in addition we assume that $p = 0$, we have the simplest case: pressure-free matter ('dust' or 'baryonic matter'). Otherwise we must specify an equation of state determining $p$ from $\mu$ and possibly other thermodynamic variables. Whatever these relations may be, we usually require that various 'energy conditions' hold: one or all of

$$\mu > 0, \; \mu + p > 0, \; \mu + 3p > 0 \tag{12}$$

and additionally demand the speed of sound $c_s$ obeys

$$0 \leq c_s^2 \leq 1 \;\Leftrightarrow\; 0 \leq dp/d\mu \leq 1.$$

### 2.1.4 The Weyl tensor

The Weyl conformal curvature tensor $C_{abcd}$ is split relative to $u^a$ into 'electric' and 'magnetic' parts:

$$E_{ac} = C_{abcd} u^b u^d \;\Rightarrow\; E^a{}_a = 0, \; E_{ab} = E_{(ab)}, \; E_{ab} u^b = 0\,, \tag{13}$$

$$H_{ac} = \frac{1}{2} \eta_{ab}{}^{ef} C_{efcd} u^b u^d \;\Rightarrow\; H^a{}_a = 0, \; H_{ab} = H_{(ab)}, \; H_{ab} u^b = 0\,. \tag{14}$$

These represent the 'free gravitational field', enabling gravitational action at a distance (tidal forces, gravitational waves). Together with the Ricci tensor $R_{ab}$ (determined locally at each point by the matter tensor through the EFE (1)), these quantities completely represent the space-time Riemann curvature tensor $R_{abcd}$.

### 2.1.5 Auxiliary quantities

It is useful to define some associated kinematic quantities: the vorticity vector

$$\omega^a = \frac{1}{2}\eta^{abcd}u_b\omega_{cd} \;\Rightarrow\; \omega^a u_a = 0,\; \omega^a\omega_{ab} = 0\,, \tag{15}$$

the magnitudes

$$\omega^2 = \frac{1}{2}\omega^{ab}\omega_{ab} \geq 0\,,\; \sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab} \geq 0\,, \tag{16}$$

and the average length scale $\ell$ determined by

$$\dot{\ell}/\ell = \frac{1}{3}\Theta\,. \tag{17}$$

Further it is helpful to define particular spatial gradients orthogonal to $u^a$, characterizing the inhomogeneity of space-time:

$$X_a = \hat{\nabla}_a\mu,\; Y_a = \hat{\nabla}_a p,\; Z_a = \hat{\nabla}_a\Theta\,. \tag{18}$$

These satisfy the important identity

$$\hat{\nabla}_{[a}\hat{\nabla}_{b]}\mu = 2\omega_{ab}\dot{\mu}\,. \tag{19}$$

The latter shows that if $\omega_{ab}\dot{\mu} \neq 0$ in an open set then $X_a \neq 0$ there.

## *2.2 Equations*

There are three sets of equations to be considered, resulting from the EFE (1).

### 2.2.1 The Ricci identity

The first set arise from the *Ricci identity* for the vector field $u^a$, i.e.

$$u^a{}_{;bc} - u^a{}_{;cb} = R_d{}^a{}_{bc}u^d\,.$$

We obtain three propagation equations and three constraint equations. The *propagation equations* are,

1. The Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a} + \frac{1}{2}\kappa(\mu + 3p) = 0\,, \tag{20}$$

which is the basic equation of gravitational attraction,

2. The vorticity propagation equation

$$h^f{}_e(\ell^2\omega^e)^{\cdot} = \ell^2\sigma^f{}_d\omega^d + \ell^2\frac{1}{2}\eta^{fcbd}u_c\dot{u}_{b;d} \tag{21}$$

showing how vorticity conservation follows if there is a perfect fluid with acceleration potential,

3. The shear propagation equation

$$\begin{aligned} h_a{}^f h_b{}^g(\ell^{-2}(\ell^2\sigma_{fg})^{\cdot} - \dot{u}_{(f;g)}) - \dot{u}_a\dot{u}_b + \omega_a\omega_b + \sigma_a{}^f\sigma_{fg} + \\ +h_{ab}(-\frac{1}{3}\omega^2 + \frac{2}{3}\sigma^2 + \dot{u}^a{}_{;a}) - \frac{1}{2}\kappa\pi_{ab} + E_{ab} = 0, \end{aligned} \tag{22}$$

showing how $E_{ab}$ induces shear.

The *constraint equations* are,

1. The $(0,\nu)$ equations

$$h^{ab}(\omega_b{}^c{}_{;d}h^d_c - \sigma_b{}^c{}_{;d}h^d_c + \frac{2}{3}\Theta_{,b}) + (\omega^a{}_b + \sigma^a{}_b)\dot{u}^b = \kappa q^a \ , \tag{23}$$

2. The vorticity divergence identity

$$\omega^a{}_{;b}h^b{}_a = \omega^a\dot{u}_a \ , \tag{24}$$

3. The $H_{ab}$ equation

$$H_{ad} = 2\dot{u}_{(a}\omega_{d)} - h_a{}^t h_d{}^s(\omega_{(t}{}^{b;c} + \sigma_{(t}{}^{b;c})\eta_{s)fbc}u^f \ . \tag{25}$$

### 2.2.2 The contracted Bianchi identities

The second set of equations arise from the *contracted Bianchi identities* (2). We obtain one propagation equation:

$$\dot{\mu} + (\mu + p)\Theta = 0\,, \tag{26}$$

the energy conservation equation, and one constraint equation:

$$(\mu + p)\dot{u}_a + h_a{}^c p_{,c} = 0\,, \tag{27}$$

the momentum conservation equation, where for simplicity we have given only the perfect fluid form.

### 2.2.3 The other Bianchi identities

If one attains a consistent solution to the equations given so far, that is all one requires. However often it is useful to additionally explicitly consider the integrability conditions for the equations listed so far. These are the *Bianchi identities*

$$R_{ab[cd;e]} = 0\,.$$

Double contraction gives (2), already considered. Apart from these equations, the full Bianchi identities give two further propagation equations and two constraint equations, which are similar in form to Maxwell's equations.

The *propagation equations* are,

$$\begin{aligned} h^m{}_a h^t{}_c \dot{E}^{ac} + J^{mt} - 2H_a{}^{(t}\eta^{m)bpq}u_b\dot{u}_p + h^{mt}\sigma^{ab}E_{ab} + \\ +\Theta E^{mt} - 3E_s{}^{(m}\sigma^{t)s} - E_s{}^{(m}\omega^{t)s} = -\frac{1}{2}(\mu+p)\sigma^{tm}\,, \end{aligned} \tag{28}$$

the '$\dot{E}$' equation, and

$$\begin{aligned} h^m{}_a h^t{}_c \dot{H}^{ac} - I^{mt} + 2E_a{}^{(t}\eta^{m)bpq}u_b\dot{u}_p + h^{mt}\sigma^{ab}H_{ab} + \\ +\Theta H^{mt} - 3H_s{}^{(m}\sigma^{t)s} - H_s{}^{(m}\omega^{t)s} = 0 \end{aligned} \tag{29}$$

the '$\dot{H}$' equation, where again we have given only the perfect fluid form, and we have defined

$$J^{mt} = h_a{}^{(m}\eta^{t)rsd}u_r H^a{}_{s;d} = \text{`}curl\ H\text{'}\,,$$

$$I^{mt} = h_a{}^{(m}\eta^{t)rsd}u_r E^a{}_{s;d} = \text{`}curl\ E\text{'}\,.$$

The *constraint equations* are

$$h^t{}_a E^{as}{}_{;d}h^d{}_s - \eta^{tbpq}u_b\sigma^d{}_p H_{qd} + 3H^t{}_s\omega^s = \frac{1}{3}h^{tb}\mu_{;b}\,, \tag{30}$$

the 'div E' equation, and

$$h^t{}_a H^{as}{}_{;d}h^d{}_s + \eta^{tbpq}u_b\sigma^d{}_p E_{qd} - 3E^t{}_s\omega^s = (\mu+p)\omega^t\,, \tag{31}$$

the 'div H' equation.

### *2.3 The set of equations*

Altogether we have six propagation equations and six constraint equations; considered as a set of evolution equations for the covariant variables, they are a first-order system of equations. This set is determinate once the fluid equations of state are given; together they then form a complete set of equations that we can regard as an infinite dimensional dynamical system (the system closes up, but is essentially infinite dimensional because of the spatial derivatives that occur).

Useful solutions are defined by considering appropriate restrictions on the kinematic quantities, Weyl tensor, or space-time geometry for a specified plausible matter content. In many cases these define a finite dimensional subset of the full system. Given such restrictions,

(a) we need to check *consistency* of the constraints with the evolution equations. It is believed that they are *generally consistent*, i.e. they are consistent if no restrictions are placed on their evolution other than implied by the evolution equations (this has not been proved, but is very plausible). Once we impose further restrictions, they may or may not be consistent. This is what we have to investigate.

(b) we need to understand the *dynamical evolution* that results, particularly fixed points, attractors, etc., in terms of suitable variables,

(c) we particularly seek to determine and characterize *involutive subsets* of the space of space-times: that is regions that are mapped into themselves by the dynamical evolution of the system, and so are left invariant by that evolution.

As far as possible we aim to do this for the exact equations. We are also concerned with

(d) *linearization* of the equations about known simple solutions, and determination of properties of the resulting linearized solutions, in particular considering whether they accurately represent the behaviour of the full non-linear theory in a neighborhood of the background solution (the issue of *linearization stability*).

The idea is to relate the different models, if possible by determining the dynamic flows in the state space of models.

## 3 Classification by symmetries

Symmetries of a space or a space-time (generically, 'space') are transformations of the space into itself that leave the metric tensor and all physical and geometrical properties invariant. We deal here only with continuous symmetries, characterized by a continuous group of transformations and associated vector fields (Eisenhart 1933).

### 3.1 *Killing vectors*

A space or space-time *symmetry* or *isometry* is a transformation that drags the metric into itself. The generating vector field $\xi_i$ is called a *Killing vector (field)* (or 'KV'), and obeys Killing's equations,

$$(L_\xi g)_{ij} = 0 \Leftrightarrow \xi_{(i;j)} = 0 \Leftrightarrow \xi_{i;j} = -\xi_{j;i} \tag{32}$$

where $L_X$ is the *Lie derivative*. By the Ricci identity for the KV, this implies the curvature equation:

$$\xi_{i;jk} = R_{ijkl}\xi^l \tag{33}$$

and so the infinite series of further equations that follows by taking covariant derivatives of this one, e.g.

$$\xi_{i;jkm} = R_{ijkl;m}\xi^l + R_{ijk}{}^l\xi_{l;m} \tag{34}$$

The Killing vector fields form a Lie algebra with a basis $\xi_a$ $(a = 1, 2, .., r)$ with components $\xi^i_a$ with respect to a local coordinate basis where a,b,c label the KV basis, i j k the coordinate components, $r \leq \frac{1}{2}n(n-1)$ is the dimension of the algebra. Any KV can be written in terms of this basis, with *constant coefficients*. Hence: if we take the commutator $[\xi_a, \xi_b]$ of two of the basis KV's, this is also a KV, and so can be written in terms of its components relative to the Killing vector basis, which will be constants. We can write the constants as $C^c{}_{ab}$, obtaining

$$[\xi_a, \xi_b] = C^c{}_{ab}\,\xi_c, \;\; C^a{}_{bc} = C^a{}_{[bc]}\,. \tag{35}$$

By the Jacobi identities for the basis vectors, these structure constants must satisfy

$$C^d{}_{s[c}C^s{}_{ab]} = 0\,. \tag{36}$$

These are the integrability conditions that must be satisfied in order that the Lie Algebra exist in a consistent way. The transformations generated by the Lie Algebra form a Lie group (Eisenhart 1933, Cohn 1961) of the same dimension.

*Arbitrariness of the basis*: we can change the basis of KV's in the usual way;

$$\xi_{a'} = \Lambda_{a'}{}^{a}\xi_{a} \;\Leftrightarrow\; \xi^{i}_{a'} = \Lambda_{a'}{}^{a}\xi^{i}_{a} \tag{37}$$

where the $\Lambda_{a'}{}^{a}$ are constants with $det(\Lambda_{a'}{}^{a}) \neq 0$, so unique inverse matrices $\Lambda^{a'a}$ exist. Then the structure constants transform as tensors:

$$C^{c'}{}_{a'b'} = C^{c}{}_{ab}\Lambda^{c'}{}_{c}\Lambda_{a'}{}^{a}\Lambda_{b'}{}^{b}\,. \tag{38}$$

Thus the (non)-equivalence of two Lie Algebras is not obvious, as they may be given in quite different bases.

### *3.2 Groups of isometries*

The isometries of a space of dimension $n$ must be a group, as the identity is an isometry, the inverse of an isometry is an isometry, and the composition of two isometries is an isometry. Continuous isometries are generated by the Lie Algebra of Killing Vector fields. The group structure is determined locally by the Lie algebra, in turn characterized by the structure constants (Cohn, 1961). The action of the group is characterized by the nature of its orbits in space; this is only partially determined by the group structure (indeed the same group can act as a space-time symmetry group in quite different ways).

#### 3.2.1 Dimensionality of groups and orbits

Most spaces have no Killing vectors, but special spaces (with symmetries) have some. The group action defines orbits in the space where it acts, and the dimensionality of these orbits determines the kind of symmetry that is present.

The *orbit* of a point $p$ is the set of all points into which $p$ can be moved by the action of the isometries of a space. Orbits are necessarily homogeneous (all physical quantities are the same at each point). An *invariant variety* is a set of points moved into itself by the group. This will be bigger than (or equal to) all orbits it contains. The orbits are necessarily invariant varieties; indeed they are sometimes called *minimum invariant varieties*, because they are the smallest subspaces that are always moved into themselves by all the isometries in the group.

*Fixed points* of groups of isometries are those points which are left invariant by the isometries (thus the orbit of such a point is just the point itself). These are the points where all Killing vectors vanish, so the dimension of the Lie algebra is zero here (however the derivatives of the Killing vectors there are non-zero; the Killing vectors generate isotropies about these points).

*General points* are those where the dimension of the space spanned by the Killing vectors (that is, the dimension of the orbit through the point) takes the value it has almost everywhere; *special points* are those where it has a lower dimension (e.g. fixed points). Consequently the dimension of the orbits through special points is lower than that of orbits through general points. The dimension of the algebra is the same at each point of an orbit, because of the equivalence of the group action at all points on each orbit.

The group is *transitive on a surface* $S$ (of whatever dimension) if it can move any point of $S$ into any other point of $S$. Orbits are the largest surfaces through each point on which the group is transitive; they are therefore sometimes referred to as *surfaces of transitivity*. We define their dimension as follows, and determine limits from the maximal possible initial data for Killing vectors:

*dim surface of transitivity* $= s =$ dim minimum invariant varieties, where in a space of dimension $n$, $s \leq n$ .

At each point we can also consider the dimension of the isotropy group (the group of isometries leaving that point fixed), generated by all those Killing vectors that vanish at that point:

*dim of isotropy group* $= q$, where $q \leq 1/2n(n-1)$.

The *dimension $r$ of the group of symmetries* of a space of dimension $n$ is $r = s + q$ (translations plus rotations). From the above limits , $0 \leq r \leq n + (1/2)n(n-1) = (1/2)n(n+1)$ (the maximal number of translations and of rotations). This shows the Lie algebra of KVs is finite dimensional.

*Maximal dimensions*: If $r = 1/2n(n+1)$ we have a space(time) of constant curvature (maximal symmetry for a space of dimension $n$). In this case,

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{39}$$

with $K$ constant; and $K$ necessarily *is* constant if this equation is true and $n \geq 3$. One can't get $q = (1/2)n(n-1) - 1$ so $r \neq (1/2)n(n+1) - 1$.

A group is *simply transitive* if $r = s \Leftrightarrow q = 0$ (no redundancy: dimensionality of group of isometries is just sufficient to move each point in a surface of transitivity into each other point). There is no continuous isotropy group.

A group is *multiply transitive* if $r > s \Leftrightarrow q > 0$ (there is redundancy in that the dimension of the group of isometries is larger than is needed to move each point in an orbit into each other point). There exist non-trivial isotropies.

### *3.3 Classification of cosmological symmetries*

For a cosmological model, because space-time is 4-dimensional, the possibilities for dimension of the surface of transitivity are $s = 0, 1, 2, 3, 4$. As to isotropy, we assume $(\mu + p) \neq 0$; then $q = 3$, $1$, or $0$ because $u^a$ is invariant and so the isotropy group at each point has to be a sub-group of the rotations acting orthogonally to $u^a$ (and there is no 2-d subgroup of $O(3)$.) The dimension $q$ of the isotropy group can vary over the space (but not over an orbit): it can be greater at special points (e.g. an axis centre of symmetry) where the dimension $s$ of the orbit is less, but $r$ (the dimension of the total symmetry group) must stay the same everywhere. Thus the possibilities for isotropy at a general point are,

**a) Isotropic**: $q = 3$, the Weyl tensor vanishes, kinematic quantities vanish except $\Theta$. All observations (at every point) are isotropic. This is the RW family of geometries;

**b) Local Rotational Symmetry** ('LRS'): $q = 1$, the Weyl tensor is type D, kinematic quantities are rotationally symmetric about a preferred spatial direction. All observations at every general point are rotationally symmetric about this direction. All metrics are known in the case of dust (Ellis 1967) and a perfect fluid (Stewart & Ellis, 1968, see also van Elst and Ellis 1996).

**c) Anisotropic**: $q = 0$; there are no rotational symmetries. Observations in each direction are different from observations in each other direction.

Putting this together with the possibilities for the dimensions of the surfaces of transitivity, we have the following possibilities [See Table 1].

## 4 Bianchi Universes ($s = 3$)

These are the models in which there is a simply transitive group $G_3$ of isometries transitive on spacelike surfaces, so they are spatially homogeneous. There is only

| Dim invariant variety → <br> Dimension Isotropy Group ↓ | s=2 <br> inhomogeneous | s=3 <br> spatially homogeneous | s=4 <br> space-time homogeneous |
| --- | --- | --- | --- |
| q = 0 <br> aniso-tropic | generic metric form known. Spatially self-similar, Abelian G_2 on 2-d spacelike surfaces, non-abelian G_2 | Bianchi: orthogonal, tilted | Osvath/Kerr |
| q = 1 <br> LRS | Bondi-Tolman family | Kantowski-Sachs, LRS Bianchi | Godel |
| q = 3 <br> isotropic | none (can't happen) | Friedmann | Einstein static |
| | two non-ignorable coordinates no redshift | one non-ignorable coordinate | algebraic EFE |

| Dim invariant variety | s=0 | s=1 |
| --- | --- | --- |
| Inhomogeneous, No Isotropy Group | Szekeres-Szafron, Stephani-Barnes, Oleson type N <br><br> The real universe! | General metric form independent of one coord; KV h.s.o., not h.s.o |

Table 1: Classification of cosmological models ($\mu + p > 0$) by isotropy and homogeneity (see Ellis 1967).

one essential dynamical coordinate, and the EFE reduce to ordinary differential equations, because the inhomogeneous degrees of freedom have been 'frozen out'. They are thus quite special in geometric terms; nevertheless they form a rich set of models where one can study the exact dynamics of the full non-linear field equations. The solutions to the field equations will depend on the matter in the space-time. In the case of a fluid (with uniquely defined flow lines), we have two different kinds of models:

*Orthogonal models*, with the fluid flow lines orthogonal to the surfaces of homogeneity (Ellis and MacCallum 1969);

*Tilted models*, with the fluid flow lines not orthogonal to the surfaces of homogeneity; the fluid velocity vector components enter as further variables (King and Ellis 1973, see also Collins and Ellis 1979).

Rotating models must be tilted, and are much more complex than non-rotating models.

### *4.1 Constructing Bianchi universes*

There are essentially three direct ways of constructing them, all based on properties of a triad of vectors $\mathbf{e}_\alpha$ that commute with the basis of Killing vectors $\xi_\beta$. Thus these approaches does not directly relate to the variables introduced in the previous section, although they will be important in understanding the Bianchi models.

The *first approach* (Taub 1951, Heckmann and Schücking 1962) puts all the time variation in the metric components:

$$ds^2 = -dt^2 + \gamma_{\alpha\beta}(t)(e^\alpha{}_i(x^\nu)dx^i)(e^\beta{}_j(x^\mu)dx^j) \tag{40}$$

where $e^\alpha{}_i(x^\nu)$ are 1-forms inverse to the spatial vector triad $e_\alpha{}^i(x^\mu)$, which have the same commutators $C^\alpha{}_{\beta\gamma}$ $(\alpha,\beta,\gamma,.. = 1,2,3)$ as the structure constants of the group of isometries and commute with the unit normal vector $e_0$ to the surfaces of homogeneity; that is, $e_\alpha = e_\alpha{}^i(\partial/\partial x^i)$, $e_0 = (\partial/\partial t)$ obey

$$[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta}e_\gamma,\ [e_0, e_\alpha] = 0\,. \tag{41}$$

One can classify the Lie Algebra structure (following Schücking) by defining

$$C^\alpha{}_{\beta\gamma}\epsilon^{\beta\gamma\delta} = n^{\alpha\delta} + \epsilon^{\alpha\delta\kappa}a_\kappa \tag{42}$$

where $n^{\alpha\beta} = n^{(\alpha\beta)}$, $a_\gamma = C^\gamma{}_{\alpha\gamma}$. Then the Jacobi Identities (36) for these vectors are

$$n^{\alpha\beta} a_\beta = 0 \tag{43}$$

We define two major classes of structure constants (and so Lie Algebras):

**Class A**: $a_\alpha = 0$,
**Class B**: $a_\alpha \neq 0$.

One can diagonalise $n_{\alpha\beta}$ in both cases by suitable choice of basis, and choose $a_\alpha$ in the 1-direction. Most of the non-zero constants (represented as constant components of $n^{\alpha\beta}$ and $a_\alpha$) can be normalised to $\pm 1$ by change of basis (37), the structure constants transforming according to (38) (and so $n^{\alpha\beta}$ and $a_\alpha$ transforming as tensors). The EFE (1) become ordinary differential equations for $\gamma_{\alpha\beta}(t)$. We deal directly with these equations, without introducing the Weyl tensor components as additional variables (so we do not explicitly consider the full set of Bianchi identities in this approach; rather they are identities that will automatically be satisfied once the EFE are satisfied).

The *second approach* (Ellis and MacCallum 1969) uses an orthonormal tetrad, so the metric components $g_{ab}$ are constants, putting all the time variation in the commutators of the basis vectors. In this case we have an orthonormal basis $\mathbf{e}_a$ $(a = 0, 1, 2, 3)$ such that

$$[e_a, e_b] = \gamma^c{}_{ab}(t) e_c \,. \tag{44}$$

The spatial commutator functions $\gamma^\alpha{}_{\beta\gamma}(t)$, which can be represented analogously to (47) above by a time-dependent matrix $n^{\alpha\beta}(t)$ and vector $a_\alpha(t)$, are equivalent to the structure constants $C^\alpha{}_{\beta\gamma}$ of the symmetry group at each point (i.e. they can be brought to the canonical forms of the $C^\alpha{}_{\beta\gamma}$ at that any by a suitable change of basis; however the transformation to do so is different at each point and at each time). The commutators $\gamma^a{}_{bc}(t)$, together with the matter variables, are then treated as the dynamical variables. The EFE (1) are first order equations for these quantities, supplemented by the Jacobi identities for the basis vectors which are also first order equations for the commutators.

The third approach is based on the automorphism group of the symmetry group. We will not consider it further here.

## 5 Dynamical systems approach

The most illuminating dynamical systems description of Bianchi models is based on the use of orthonormal tetrads, and is examined in detail in a forthcoming book (Wainwright and Ellis 1996). The variables used are essentially the commutator coefficients mentioned above, but rescaled by a common time dependent factor[1].

### 5.1 The reduced differential equations

The basic idea (Collins 1971, Wainwright 1988) is to write the Einstein field equations in a way that enables one to study the evolution of the various physical and geometrical quantities *relative to the overall rate of expansion of the universe*, as described by the rate of expansion scalar $\theta = u^a_{;a}$, or equivalently *the Hubble variable* $H$:

$$H = \tfrac{1}{3}\theta. \tag{45}$$

We consider here non-tilted fluids, where the 4-velocity $\mathbf{u}$ is orthogonal to the group orbits and $t$ is a time variable which is constant on the group orbits, so that $\mathbf{u} = \frac{\partial}{\partial t}$. Let $\{\mathbf{e}_a\}$ be a group invariant orthonormal frame, with $\mathbf{e}_0 = \mathbf{u}$. We use the commutation functions $\gamma^c_{ab}$ associated with the frame $\{\mathbf{e}_a\}$: as the basic gravitational field variables. The $\gamma^c_{ab}$ are constant on the group orbits and can thus be regarded as a function of the time variable $t$: $\gamma^c_{ab} = \gamma^c_{ab}(t)$. Since $\mathbf{e}_0$ is normal to the group orbits, the non-zero commutation functions are

$$(\gamma^c{}_{ab}) = (H, \sigma_{\alpha\beta}, \Omega_\alpha, n_{\alpha\beta}, a_\alpha)\,, \tag{46}$$

where $H(t)$, $\sigma_{\alpha\beta}(t)$ are the expansion and shear of the normal flow lines, $\Omega_\alpha(t)$ is the rate of rotation of the spatial tetrad vectors relative to a parallel propagated basis along the fluid flow lines, and $n_{\alpha\beta}(t)$, $a_\alpha(t)$ represent the purely spatial commutators through the equation

$$\gamma^\alpha{}_{\beta\gamma}\epsilon^{\beta\gamma\delta} = n^{\alpha\delta}(t) + \epsilon^{\alpha\delta\kappa}a_\kappa(t) \tag{47}$$

(cf. (42)). At this stage the remaining freedom in the choice of orthonormal frame needs to be eliminated by specifying the variables $\Omega_\alpha$ implicitly or explicitly (for example by specifying them as functions of the $\sigma_{\alpha\beta}$). This also simplifies the other quantities (for example choice of a shear eigenframe will result in the

[1]The following is adapted from notes by J. Wainwright

tensor $\sigma_{\alpha\beta}$ being represented by two diagonal terms). This leads to a reduced set of variables, consisting of $H$ and the remaining commutation functions, which we denote symbolically by

$$\mathbf{x} = (\gamma^c{}_{ab}|_{reduced}). \tag{48}$$

The physical state of the model is thus described by the vector $(H, \mathbf{x})$. The details of this reduction differ for the class A and B models, and in the latter case there is an algebraic constraint of the form

$$g(\mathbf{x}) = 0\,, \tag{49}$$

where $g$ is a homogeneous polynomial.

The idea is now to normalize $\mathbf{x}$ with the Hubble variable $H$. We denote the resulting variables by a vector $\mathbf{y} \in \mathbb{R}^n$, and write:

$$\mathbf{y} = \frac{\mathbf{x}}{H}. \tag{50}$$

These new variables are *dimensionless*, and will be referred to as *expansion-normalized variables*. It is clear that each dimensionless state $\mathbf{y}$ determines a 1-parameter family of physical states $(\mathbf{x}, H)$. The evolution equations for the $\gamma^c{}_{ab}$ lead to evolution equations for $H$ and $\mathbf{x}$ and hence for $\mathbf{y}$. In deriving the evolution equations for $\mathbf{y}$ from those for $\mathbf{x}$, the *deceleration parameter* $q$ plays an important role. The Hubble variable $H$ can be used to define a scale factor $\ell$, according to

$$H = \frac{\dot{\ell}}{\ell}, \tag{51}$$

where $\cdot$ denotes differentiation with respect to $t$. The deceleration parameter is then defined by

$$\dot{H} = -(1+q)H^2. \tag{52}$$

In order that the evolution equations define a flow, it is necessary, in conjunction with the rescaling (50) to introduce a *dimensionless time variable* $\tau$ according to

$$\ell = \ell_0 e^{\tau}, \tag{53}$$

where $\ell_0$ is the value of the scale factor at some arbitrary reference time. Since $\ell$ assumes values $0 < \ell < +\infty$ in an ever-expanding model, $\tau$ assumes all real values, with $\tau \to -\infty$ at the initial singularity and $\tau \to +\infty$ at late times. It follows from equations (51) and (53) that

$$\frac{dt}{d\tau} = \frac{1}{H}, \tag{54}$$

and the evolution equation (52) for $H$ can be written

$$\frac{dH}{d\tau} = -(1+q)H. \tag{55}$$

Since the right hand side of the evolution equations for the $\gamma^c_{ab}$ are homogeneous of degree 2 in the $\gamma^c{}_{ab}$ the change (54) of the time variable results in $H$ canceling out of the evolution equation for $\mathbf{y}$, yielding an autonomous DE:

$$\frac{d\mathbf{y}}{d\tau} = \mathbf{f}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n. \tag{56}$$

The constraint $g(\mathbf{x}) = 0$ translates into a constraint

$$g(\mathbf{y}) = 0, \tag{57}$$

which is preserved by the DE. The functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are polynomial functions in $\mathbf{y}$. An essential feature of this process is that the evolution equation for $H$, namely (55), decouples from the remaining equations (56) and (57). In other words, the DE (56) describes the evolution of the non-tilted Bianchi cosmologies, the transformation (5.5) essentially scaling away the effects of the overall expansion. An important consequence is that the new variables are bounded near the initial singularity.

## *5.2 Cosmological dynamical systems*

### 5.2.1 Invariant sets and limit sets

The first step in the analysis is to formulate the field equations, using expansion-normalized variables, as a DE (56) in $\mathbb{R}^n$, possibly subject to a constraint (57). Since $\tau$ assumes all real values (for models which expand indefinitely), the solutions of (56) are defined for all $\tau$ and hence define a *flow* $\{\phi_\tau\}$ on $\mathbb{R}^n$. The evolution of the cosmological models can thus be analyzed by studying the orbits of this flow in the physical region of state space, which is a subset of $\mathbb{R}^n$ defined by the requirement that the energy density be non-negative, i.e.

$$\Omega(\mathbf{y}) = \frac{\mu}{3H^2} \geq 0 \tag{58}$$

where the density parameter $\Omega$ is a dimensionless measure of the matter density $\mu$.

The *vacuum boundary*, defined by $\Omega(\mathbf{y}) = 0$, describes the evolution of vacuum Bianchi models, and is an invariant set which plays an important role in the qualitative analysis because vacuum models can be asymptotic states for perfect fluid models near the big-bang or at late times. There are other invariant sets which are also specified by simple restrictions on $\mathbf{y}$ which play a special role: the subsets representing each Bianchi type, and the subsets representing higher symmetry models, specifically the FL models and the LRS Bianchi models.

It is desirable that the dimensionless state space $D$ in $\mathbb{R}^n$ is a compact set. In this case each orbit will have a non-empty $\alpha$-limit set and $\omega$-limit set, and hence there will exist a past attractor and a future attractor in state space. When using expansion-normalized variables, compactness of the state space has a direct physical meaning for ever-expanding models: if the state space is compact then at the big-bang no physical or geometrical quantity diverges more rapidly than the appropriate power of $H$, and at late times no such quantity tends to zero less rapidly than the appropriate power of $H$. This will happen for many models; however the state space for Bianchi $\text{VII}_0$ and VIII models is non-compact. This lack of compactness manifests itself in the behaviour of the Weyl tensor at late times.

### 5.2.2 Equilibrium points and self-similar cosmologies

Each ordinary orbit in the dimensionless state space corresponds to a one-parameter family of physical universes, which are conformally related by a constant rescaling of the metric. On the other hand, for an equilibrium point $\mathbf{y}^*$ of the DE (56) (which satisfies $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$), the deceleration parameter $q$ is a constant, i.e. $q(\mathbf{y}^*) = q^*$, and we find

$$H(\tau) = H_0 e^{(1+q^*)\tau}.$$

In this case, however, the parameter $H_0$ is no longer essential, since it can be set to unity by a translation of $\tau$, $\tau \to \tau +$ constant; then (54) implies that

$$Ht = \frac{1}{1+q^*}, \tag{59}$$

so that by (48) and (50) the commutation functions are of the form (constant) $\times\, t^{-1}$. It follows that the resulting cosmological model is self-similar. Thus, *to each equilibrium point of the DE (56) there corresponds a unique self-similar cosmological model.* In such a model the physical states at different times differ only by an overall change in the length scale. Such models are expanding, but in such a way that their dimensionless state does not change. They include the flat FL model ($\Omega = 1$) and the Milne model ($\Omega = 0$). All vacuum and non-tilted perfect fluid self-similar Bianchi solutions have been given by Hsu and Wainwright (1986).

The equilibrium points determine the asymptotic behaviour of other more general models. If the $\alpha$-limit set of a point $\mathbf{y}$ is an equilibrium point $\mathbf{y}^*$, then the orbit through $\mathbf{y}$ approaches $\mathbf{y}^*$ as $\tau \to -\infty$. The physical interpretation is that the self-similar model which corresponds to $\mathbf{y}^*$ approximates the dynamics of the model with initial state $\mathbf{y}$, as $\tau \to -\infty$. This model is *asymptotically self-similar into the past.* A similar interpretation holds if the $\omega$-limit set is an equilibrium point. The term *asymptotically selfsimilar* without a qualifier means that the model has this property into the past and into the future. In this case the orbit that describes the model will be *heteroclinic* (i.e. joins two equilibrium points).

Equilibrium points also influence the intermediate evolution by determining finite heteroclinic sequences which join the past attractor to the future attractor. The intermediate equilibrium points in the sequence determine quasi-equilibrium epochs, which may be important from an observational point of view. In this case an anisotropic model can spend an arbitrarily large time in an $\epsilon$-neighbourhood arbitrarily close to a FL equilibrium point; and hence can for practical purposes by observationally indistinguishable from it, even though its very early and very late behaviour will both be completely different.

Many phase planes can be constructed explicitly. The reader is referred to Wainright and Ellis (1996) for a comprehensive presentation and survey of results attained so far.

### *5.3 Relation to full state space*

The symmetric spaces, such as the Bianchi universes, form finite dimensional subsets of the full dynamical system, defining involutive subsets of the full state space of solutions. There are also involutive subspaces that are infinite dimensional, some of which are discussed in the next section. The challenge is to characterise them and to relate them to the finite dimensional subspaces, such as those associated with Bianchi models.

## 6 Other involutive subspaces of state space

We look at some of these infinite dimensional subspaces here, and then briefly comment on the relation to the finite dimensional subspaces in the following section.

### 6.1 Pressure-free matter ('dust')

A particularly useful dynamical restriction is

$$p = 0 = q_a = \pi_{ab}$$

so the matter (often described as 'baryonic') is represented only by its 4-velocity $u^a$ and its density $\mu > 0$.

In this case momentum conservation shows that $\dot{u}_a = 0$: the matter moves geodesically (as expected from the equivalence principle), and all equations simplify considerably. This is the case of *pure gravitation*: it separates out the (non-linear) gravitational effects from all the fluid dynamic effects. It is a very large involutive subspace.

### 6.2 Irrotational flow

If we have a barotropic perfect fluid:

$$q_q = \pi_{ab} = 0, \; p = p(\mu) \;\Rightarrow \eta^{acd}\dot{u}_{c;d} = 0$$

then $\omega = 0$ is involutive, i.e.

$$\omega^a = 0 \;\Rightarrow\; \dot{\omega}^a = 0$$

follows from the vorticity conservation equations (and this is true also in the special case $p = 0$), see (Ehlers 1961, 1993; Ellis 1973). In such flows,

**1.** The fluid flow is hypersurface orthogonal, as there exists a cosmic time function $t$ such that $u_a = -g(x^b)t_{,a}$,

**2.** The metric of the orthogonal 3-spaces is $h_{ab}$,

**3.** The Ricci tensor of these 3-spaces is given by

$$\begin{aligned} {}^3R_{ab} &= h_a{}^f h_b{}^g \left[\dot{u}_{(f;g)} - \ell^{-3}(\ell^3\sigma_{fg})^{\cdot}\right] + \dot{u}_a\dot{u}_b + \\ &+ \frac{2}{3}(-\frac{1}{3}\Theta^2 + \sigma^2 - \frac{1}{2}\dot{u}^c{}_{;c} + \Lambda + \kappa\mu) + \kappa\pi_{ab} \end{aligned} \qquad (60)$$

and their Ricci scalar by

$$^3R = 2\{\sigma^2 - \frac{1}{3}\Theta^2 + \Lambda + \kappa\mu\} \ , \tag{61}$$

which is a generalised Friedmann equation. These equations fully determine the curvature tensor $^3R_{abcd}$ of the orthogonal 3-spaces. Provided the matter is baryonic perfect fluid, this is an involutive subspace of large dimension.

### *6.3 Irrotational dust*

Dust is a special case of a baryonic fluid, so the dust irrotational spaces form an involutive subspace which is the intersection of the two. Considering these solutions, $p = 0 \Rightarrow \dot{u}_a = 0$ and $\omega^a = 0$. Then the non-trivial (exact) evolution equations of Section 1.2 are,

$$\dot{\mu} + \mu\Theta = 0 \, , \tag{62}$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + \frac{1}{2}\kappa\mu = 0 \, , \tag{63}$$

$$\dot{\sigma}_{ab} + \sigma_a{}^f\sigma_{fb} - h_{ab}\frac{2}{3}\sigma^2 + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} = 0, \tag{64}$$

$$\dot{E}^{mt} + h^{mt}\sigma^{ab}E_{ab} + \Theta E^{mt} + J^{mt} - 3E_s{}^{(m}\sigma^{t)s} = -\frac{1}{2}\mu\sigma^{tm} \, , \tag{65}$$

$$\dot{H}^{mt} + h^{mt}\sigma^{ab}H_{ab} + \Theta H^{mt} - I^{mt} - 3H_s{}^{(m}\sigma^{t)s} = 0 \, , \tag{66}$$

where $J^{mt}$ is 'curl H' and $I^{mt}$ is 'curl E'.

The constraint equations are

$$h^{ab}(-\sigma_b{}^c{}_{;d}h^d_c + \frac{2}{3}\Theta_{,b}) = 0 \, , \tag{67}$$

$$H_{ad} = -h_a{}^t h_d{}^s \sigma_{(t}{}^{b;c})\eta_{s)fbc}u^f \ , \tag{68}$$

$$h^t{}_a E^{as}{}_{;d}h^d{}_s - \eta^{tbpq}u_b\sigma^d{}_p H_{qd} = \frac{1}{3}X^t \, , \tag{69}$$

$$h^t{}_a H^{as}{}_{;d}h^d{}_s + \eta^{tbpq}u_b\sigma^d{}_p E_{qd} = 0 \, . \tag{70}$$

In general these equations are consistent (Maartens et al. 1997).

### 6.4 FL universes (RW geometry)

A particularly important involutive subspace of the irrotational dust space-times is that of the Friedmann-Lemaître ('FL') universes, based on the everywhere-isotropic Robertson-Walker ('RW') geometry. It is characterized by a perfect fluid matter tensor and the conditions

$$\omega_{ab} = \sigma_{ab} = 0 = \dot{u}^a \;\Rightarrow\; E_{ab} = H_{ab} = 0,\; X_a = Y_a = Z_a = 0\,,$$

the first conditions stating these solutions are also shear-free and hence are locally isotropic, the second that they are conformally flat, and the third that they are spatially homogeneous. It follows then that:

**1.** ${}^3R_{ab}$ is isotropic, so the 3-spaces are 3-spaces of constant curvature;

**2.** The remaining non-trivial equations are the energy equation (26), the Raychaudhuri equation (20) which now takes the form

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}\kappa(\mu + 3p) = 0\,, \tag{71}$$

and the Friedmann equation that follows from (61):

$${}^3R = -\frac{2}{3}\Theta^2 + 2\kappa\mu = \frac{6k}{\ell^2}\,, \tag{72}$$

where $k$ is a constant. Any two of these equations imply the third if $\theta \neq 0$ (the latter equation being a first integral of the other two).

**3.** From these equations, as well as finding simple exact solutions one can determine evolutionary phase planes for this family of models, see Refsdal and Stabell (1966), Madsen and Ellis (1988), and Ehlers and Rindler (1989).

### 6.5 The Shear-Free case

If $p = 0 \Rightarrow \dot{u}_a = 0$ and $\sigma_{ab} = 0$ in an open set $\mathcal{U}$ then all equation simplify in $\mathcal{U}$. In particular the vorticity equation becomes

$$h^a{}_b(\ell^2\Omega^b)^{\cdot} = 0 \;\Rightarrow\; \omega^a = \frac{\Omega^a}{\ell^2},\; (\Omega^a)^{\cdot} = 0 \tag{73}$$

and then (on using the energy conservation equation) we can integrate the Raychaudhuri equation to get a 'Friedmann equation'

$$3(\dot{\ell})^2 + \frac{2\Omega^2}{\ell^2} - \frac{M}{\ell} = E \tag{74}$$

where $M, E$ are constants. This appears to allow the avoidance of an initial singularity, as the vorticity term can dominate at early times! BUT putting $\sigma_{ab} = 0$ converts the $\dot{\sigma}_{ab}$ equation (22) into a new constraint:

$$\omega_a \omega_b + h_{ab}(-\frac{1}{3}\omega^2) = -E_{ab} \,. \tag{75}$$

This has to be consistent with the time evolution of $E_{ab}$, which now takes the form

$$h^m{}_a h^t{}_c \dot{E}^{ac} + J^{mt} + \Theta E^{mt} - E_s{}^{(m}\omega^{t)s} = 0 \,. \tag{76}$$

We must now systematically check consistency.

The *Procedure* is as follows: take the time derivatives of all new constraints that arise from our assumptions (here, (75)). If necessary, commute space and time derivatives in the resulting equations, using the Ricci identities to do so. Substitute for the time evolution terms from the evolution equations, and use Leibniz's rule to expand out the spatial derivatives. Collect terms, obtaining simplified equations without any time derivatives. The result is *either* a new constraint equation that must be satisfied if the original constraint is to be preserved in time, *or* an identity $(0 = 0)$. CONTINUE until all the constraints that arise in this way are identically conserved by the time evolution, *or* we get an inconsistency.

The result of this procedure (Ellis 1967) is that in order to be consistent, shear-free dust solutions cannot expand and rotate; in $\mathcal{U}$,

$$\omega\Theta = 0 \;\Rightarrow\; if\ \Theta \neq 0,\ then\ \omega = 0 \,. \tag{77}$$

Thus the only expanding dust solutions with vanishing shear are the FL solutions. Hence this does not offer a route to singularity avoidance (for consistency, the constant $\Omega$ in equation (74) has to vanish, so the vorticity term cannot dominate the early expansion.) The involutive subspace of irrotational dust space-times defined by this condition is just the FL subspace.

### *6.6 Silent universes: $H_{ab} = 0$.*

The evolution equations for irrotational dust, in general partial differential equations, become ordinary differential equations if $I^{mt} = 0 = J^{mt}$: with these restrictions, there are no spatial derivatives in these equations. Hence we then have what has been called a 'silent universe' —provided the constraints are satisfied initially, and are conserved by the evolution equations, each world line evolves independently of each other (this evolution being governed by o.d.e's). In this case the infinite dimensional dynamical system decomposes into the direct product of finite dimensional dynamical systems along each world line.

The simplest case is when $H_{ab} = 0$. Then the equation (66) becomes a new constraint:

$$I^{mt} = h_a{}^{(m}\eta^{t)rsd}u_r E^a{}_{s;d} = 0\,. \tag{78}$$

Is this constraint (and the other constraints) preserved along the flow lines? No they are not, as has been shown by Bonilla et al (1996) and by van Elst et al (1996)[2]. The proof is based on analysis using a tetrad that simultaneously diagonalizes $\sigma_{ab}$ and $E_{ab}$ (possible because of (70)). It is not known what the full set of consistent solutions is, that forms an involutive subset of the exact field equations; it includes Bianchi I universes and the Szekeres family of models. There may be no others.

### 6.7 $div\ H = 0$

Now consider the case of solutions with $div\ H = 0$. Equation (70) then shows

$$div\ H = 0\ \Rightarrow\ \eta^{tbpq}u_b\sigma^d{}_p E_{qd} = 0\,, \tag{79}$$

so $E_{ab}$ and $\sigma_{ab}$ can be simultaneously diagonalised. This reduces the number of variables drastically. We now need to check the consistency of the new condition, that is, to examine the consequences of the equation $(div\ H)^{\cdot} = 0$, using the same procedure as before. A consistency analysis (Maartens et al 1997) [3] shows this is consistent, even if $H \neq 0$. This is an exact result following from the full field equations, and shows consistency of these equations with the usual results of linearised theory for gravitational waves. Hence this does form an involutive subset of the full space of solutions.

These examples show how examination of the integrability conditions of the exact field equations starts to delineate allowed subspaces in the space of cosmological space-times. There is much to be done here, for example extending the above analyses to the case where $\omega_{ab} \neq 0$, or to $p = p(\mu)$.

[2] Correcting previous incorrect claims by Lesame et al

[3] Correcting Lesame et al 1996, which is erroneous because of a sign error in the equations used.

## 7 Problems and Issues

A lot of progress has been made in recent times, but many issues remain outstanding. So far, the covariant approach has not been properly tied in to the exact solutions characterized by symmetries. That is, the two main sections above have not been related properly to each other. This is an unsolved problem at the present time - the Locally Rotationally Symmetric case has been solved (van Elst and Ellis 1996) and some partial results are known in other cases. But we do not have a simple characterization of the symmetric subspaces —for example the Bianchi universes— in terms of the covariant variables.

The broader aim is an understanding of the evolution of models in the space of space-times, characterizing invariant sets, fixed points, saddle points, attractors, etc. As seen above, we can find these features in some phase planes that are sections of the full space of space-times, corresponding to families of higher-symmetry solutions or to kinematic restrictions; they then determine the nature of the evolutionary curves in those families (Wainwright and Ellis 1996). The problem is to extend this understanding to broader classes of models, and the to relation between the covariant and symmetry approaches.

Other issues that have not yet been resolved are:

(1) finding a suitable measure of probability in the full space of space-times, and in its involutive subspaces. The requirement is a natural measure that is plausible. Progress has been made in the FL sub-case, but even here is not definitive.

(2) Relating descriptions of the same space-time on different scales of description. This leads to the issue of averaging and the resulting effective (polarization) contributions to the stress tensor, arising because averaging does not commute with calculating the field equations for a given metric.

(3) Related to this is the question of definition of entropy for gravitating systems in general, and cosmological models in particular. This may be expected to imply a coarse-graining in general, and so is strongly related to the averaging question. It is an important issue in terms of its relation to the spontaneous formation of structure in the early universe.

## References


[1] M. A. G. Bonilla, M. Mars, J. M. M. Senovilla, C. F. Sopuerta and R. Vera (1996). *Phys. Rev.* D **54**, 6565.

[2] P. M. Cohn (1961). *Lie Groups.* Cambridge University Press, Cambridge.

[3] C. B. Collins (1985). *J. Math. Phys.* **26**, 2009.

[4] C. B. Collins and G. F. R. Ellis (1979). *Phys. Rep.* **56**, 63.

[5] J. Ehlers (1961). *Abh. Mainz Akad. Wiss. u. Litt.*, Mat-Nat. Kl., Nr. 11.

[6] J. Ehlers (1993). *Gen. Rel. Grav.* **25**, 1225.

[7] J. Ehlers and W. Rindler (1989). *Mon. Not. Roy. Ast. Soc.* **238**, 503.

[8] L. P. Eisenhart (1933). *Continuous groups of transformations.* Dover (reprinted).

[9] G. F. R. Ellis (1967). *J. Math. Phys.* **8**, 1171-1194 .

[10] G. F. R. Ellis (1971). In *General Relativity and Cosmology*, Proc. Int. School of Physics "Enrico Fermi" (Varenna), Course XLVII. Ed. R. K. Sachs (Academic Press), 104.

[11] G. F. R. Ellis (1973). In *Cargèse Lectures in Physics, Vol. VI*, Ed. E. Schatzman (Gordon and Breach), 1.

[12] G. F. R. Ellis and M. A. H. MacCallum (1969). *Comm. Math. Phys.* **12**, 108.

[13] O. Heckmann and E. Schücking (1962). In *Gravitation: An Introduction to current research.* Ed. L. Witten (Wiley), 438-469.

[14] A. R. King and G. F. R. Ellis (1973). *Comm. Math. Phys.* **31**, 209.

[15] L. Hsu and J. Wainwright (1986). *Class. Quant. Grav.* **3**, 1105.

[16] A. Krasinski (1993). *Physics in an Inhomogeneous Universe.* Preprint 1993/10, Applied Mathematics Department, University of Cape Town. To appear as a book, Cambridge University Press (1997).

[17] W. Lesame, G. F. R. Ellis and P. K. S. Dunsby (1996). *Phys. Rev.* D **53**, 738.

[18] R. Maartens, W. M. Lesame, and G. F. R. Ellis (1997). Consistency of dust solutions with div $H = 0$. To appear, *Phys. Rev.* D.

[19] M. S. Madsen and G. F. R. Ellis (1988). *Mon. Not. Roy. Ast. Soc.* **234**, 67.

[20] R. Stabell and S. Refsdal (1966). *Mon. Not. Roy. Ast. Soc.* **132**, 379.

[21] J. M. Stewart and G. F. R. Ellis (1968). *J. Math. Phys.* **9**, 1072-1082.

[22] A. Taub (1951). *Ann. Math.* **53**, 472.

[23] H. van Elst and G. F. R. Ellis (1996). *Class. Quant. Grav.* **13**, 1099-1127.

[24] H. van Elst, C. Uggla, W. M. Lesame, R. Maartens, and G. F. R. Ellis (1997). *Class. Quant. Grav.* (1997) **14**, 1151–1162.

[25] J. Wainwright (1988). In *Relativity Today: Proc. 2nd Hungarian relativity Workshop 1987*, Ed. Z. Perjes. (Singapore: World Scientific).

[26] J. Wainwright and G. F. R. Ellis (1996). *Dynamical Systems in Cosmology.*